# Section 1.5

# Linear and Affine Functions

One of the central themes of calculus is the approximation of nonlinear functions by linear functions, with the fundamental concept being the derivative of a function. This section will introduce the linear and affine functions which will be key to understanding derivatives in the chapters ahead.

## Linear functions

In the following, we will use the notation $f : \mathbb{R}^m \to \mathbb{R}^n$ to indicate a function whose domain is a subset of $\mathbb{R}^m$ and whose range is a subset of $\mathbb{R}^n$. In other words, $f$ takes a vector with $m$ coordinates for input and returns a vector with $n$ coordinates. For example, the function

$$f(x, y, z) = (\sin(x + y), 2x^2 + z)$$

is a function from $\mathbb{R}^3$ to $\mathbb{R}^2$.

**Definition** We say a function $L : \mathbb{R}^m \to \mathbb{R}^m$ is *linear* if (1) for any vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$,

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}), \tag{1.5.1}$$

and (2) for any vector $\mathbf{x}$ in $\mathbb{R}^m$ and scalar $a$,

$$L(a\mathbf{x}) = aL(\mathbf{x}). \tag{1.5.2}$$

**Example** Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x$. Then for any $x$ and $y$ in $\mathbb{R}$,

$$f(x + y) = 3(x + y) = 3x + 3y = f(x) + f(y),$$

and for any scalar $a$,

$$f(ax) = 3ax = af(x).$$

Thus $f$ is linear.

**Example** Suppose $L : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$L(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, 4x_2).$$

Then if $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are vectors in $\mathbb{R}^2$,

$$\begin{aligned} L(\mathbf{x} + \mathbf{y}) &= L(x_1 + y_1, x_2 + y_2) \\ &= (2(x_1 + y_1) + 3(x_2 + y_2), x_1 + y_1 - (x_2 + y_2), 4(x_2 + y_2)) \\ &= (2x_1 + 3x_2, x_1 - x_2, 4x_2) + (2y_1 + 3y_2, y_1 - y_2, 4y_2) \\ &= L(x_1, x_2) + L(y_1, y_2) \\ &= L(\mathbf{x}) + L(\mathbf{y}). \end{aligned}$$

Also, for $\mathbf{x} = (x_1, x_2)$ and any scalar $a$, we have

$$\begin{aligned} L(a\mathbf{x}) &= L(ax_1, ax_2) \\ &= (2ax_1 + 3ax_2, ax_1 - ax_2, 4ax_2) \\ &= a(2x_2 + 3x_2, x_1 - x_2, 4x_2) \\ &= aL(\mathbf{x}). \end{aligned}$$

Thus $L$ is linear.

Now suppose $L : \mathbb{R} \to \mathbb{R}$ is a linear function and let $a = L(1)$. Then for any real number $x$,

$$L(x) = L(1x) = xL(1) = ax. \qquad (1.5.3)$$

Since any function $L : \mathbb{R} \to \mathbb{R}$ defined by $L(x) = ax$, where $a$ is a scalar, is linear (see Problem 1), it follows that the only functions $L : \mathbb{R} \to \mathbb{R}$ which are linear are those of the form $L(x) = ax$ for some real number $a$. For example, $f(x) = 5x$ is a linear function, but $g(x) = \sin(x)$ is not.

Next, suppose $L : \mathbb{R}^m \to \mathbb{R}$ is linear and let $a_1 = L(\mathbf{e}_1), a_2 = L(\mathbf{e}_2), \ldots, a_m = L(\mathbf{e}_m)$. If $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ is a vector in $\mathbb{R}^m$, then we know that

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m.$$

Thus

$$\begin{aligned} L(\mathbf{x}) &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m) \\ &= L(x_1\mathbf{e}_1) + L(x_2\mathbf{e}_2) + \cdots + L(x_m\mathbf{e}_m) \\ &= x_1L(\mathbf{e}_1 + x_2L(\mathbf{e}_2) + \cdots + x_mL(\mathbf{e}_m) \qquad (1.5.4) \\ &= x_1a_1 + x_2a_2 + \cdots + x_ma_m \\ &= \mathbf{a} \cdot \mathbf{x}, \end{aligned}$$

where $a = (a_1, a_2, \ldots, a_m)$. Since for any vector $\mathbf{a}$ in $\mathbb{R}^m$, the function $L(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ is linear (see Problem 1), it follows that the only functions $L : \mathbb{R}^m \to \mathbb{R}$ which are linear are those of the form $L(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$ for some fixed vector $\mathbf{a}$ in $\mathbb{R}^m$. For example,

$$f(x, y) = (2, -3) \cdot (x, y) = 2x - 3y$$

is a linear function from $\mathbb{R}^2$ to $R$, but

$$f(x, y, z) = x^2y + \sin(z)$$

is not a linear function from $\mathbb{R}^3$ to $R$.

Now consider the general case where $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear function. Given a vector $\mathbf{x}$ in $\mathbb{R}^m$, let $L_k(\mathbf{x})$ be the $k$th coordinate of $L(\mathbf{x})$, $k = 1, 2, \ldots, n$. That is,

$$L(\mathbf{x}) = (L_1(\mathbf{x}), L_2(\mathbf{x}), \ldots, L_n(\mathbf{x})).$$

Since $L$ is linear, for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^m$ we have

$$L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),$$

or, in terms of the coordinate functions,

$$\begin{aligned}(L_1(\mathbf{x}+\mathbf{y}), L_2(\mathbf{x}+\mathbf{y}), \ldots, L_n(\mathbf{x}+\mathbf{y})) &= (L_1(\mathbf{x}), L_2(\mathbf{x}), \ldots, L_n(\mathbf{x})) \\ &\qquad + (L_1(\mathbf{y}), L_2(\mathbf{y}), \ldots, L_n(\mathbf{y})) \\ &= (L_1(\mathbf{x}) + L_1(\mathbf{y}), L_2(\mathbf{x}) + L_2(\mathbf{y}), \\ &\qquad \ldots, L_n(\mathbf{x}) + L_n(\mathbf{y})).\end{aligned}$$

Hence $L_k(\mathbf{x}+\mathbf{y}) = L_k(\mathbf{x}) + L_k(\mathbf{y})$ for $k = 1, 2, \ldots, n$. Similarly, if $\mathbf{x}$ is in $\mathbb{R}^m$ and $a$ is a scalar, then $L(a\mathbf{x}) = aL(\mathbf{x})$, so

$$\begin{aligned}(L_1(a\mathbf{x}), L_2(a\mathbf{x}), \ldots, L_n(a\mathbf{x}) &= a(L_1(\mathbf{x}), L_2(\mathbf{x}), \ldots, L_n(x)) \\ &= (aL_1(\mathbf{x}), aL_2(\mathbf{x}), \ldots, aL_n(x)).\end{aligned}$$

Hence $L_k(a\mathbf{x}) = aL_k(\mathbf{x})$ for $k = 1, 2, \ldots, n$. Thus for each $k = 1, 2, \ldots, n$, $L_k : \mathbb{R}^m \to \mathbb{R}$ is a linear function. It follows from our work above that, for each $k = 1, 2, \ldots, n$, there is a fixed vector $\mathbf{a}_k$ in $\mathbb{R}^m$ such that $L_k(x) = \mathbf{a}_k \cdot \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^m$. Hence we have

$$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}) \tag{1.5.5}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. Since any function defined as in (1.5.5) is linear (see Problem 1 again), it follows that the only linear functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ must be of this form.

**Theorem** If $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then there exist vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$ such that

$$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}) \tag{1.5.6}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$.

**Example** In a previous example, we showed that the function $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, 4x_2)$$

is linear. We can see this more easily now by noting that

$$L(x_1, x_2) = ((2, 3) \cdot (x_1, x_2), (1, -1) \cdot (x_1, x_2), (0, 4) \cdot (x_1, x_2)).$$

**Example** The function

$$f(x, y, z) = (x + y, \sin(x + y + z))$$

is not linear since it cannot be written in the form of (1.5.6). In particular, the function $f_2(x, y, z) = \sin(x+y+z)$ is not linear; from our work above, it follows that $f$ is not linear.

**Matrix notation**

We will now develop some notation to simplify working with expressions such as (1.5.6). First, we define an $n \times m$ matrix to be to be an array of real numbers with $n$ rows and $m$ columns. For example,

$$M = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$$

is a $3 \times 2$ matrix. Next, we will identify a vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ in $\mathbb{R}^m$ with the $m \times 1$ matrix

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix},$$

which is called a *column vector*. Now define the product $M\mathbf{x}$ of an $n \times m$ matrix $M$ with an $m \times 1$ column vector $\mathbf{x}$ to be the $n \times 1$ column vector whose $k$th entry, $k = 1, 2, \ldots, n$, is the dot product of the $k$th row of $M$ with $\mathbf{x}$. For example,

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+3 \\ 2-1 \\ 0+4 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix}.$$

In fact, for any vector $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$,

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ x_1 - x_2 \\ 4x_2 \end{bmatrix}.$$

In other words, if we let

$$L(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2, 4x_2),$$

as in a previous example, then, using column vectors, we could write

$$L(x_1, x_2) = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In general, consider a linear function $L : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}) \tag{1.5.7}$$

for some vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in $\mathbb{R}^m$. If we let $M$ be the $n \times m$ matrix whose $k$th row is $\mathbf{a}_k$, $k = 1, 2, \ldots, n$, then

$$L(\mathbf{x}) = M\mathbf{x} \tag{1.5.8}$$

for any $\mathbf{x}$ in $\mathbb{R}^m$. Now, from our work above,

$$\mathbf{a}_k = (L_k(\mathbf{e}_1), L_k(\mathbf{e}_2), \ldots, L_k(\mathbf{e}_m)), \tag{1.5.9}$$

which means that the $j$th column of $M$ is

$$\begin{bmatrix} L_1(\mathbf{e}_j) \\ L_2(\mathbf{e}_j) \\ \vdots \\ L_n(\mathbf{e}_j) \end{bmatrix}, \tag{1.5.10}$$

$j = 1, 2, \ldots, m$. But (1.5.10) is just $L(\mathbf{e}_j)$ written as a column vector. Hence $M$ is the matrix whose columns are given by the column vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_m)$.

**Theorem** Suppose $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear function and $M$ is the $n \times m$ matrix whose $j$th column is $L(\mathbf{e}_j)$, $j = 1, 2, \ldots, m$. Then for any vector $\mathbf{x}$ in $\mathbb{R}^m$,

$$L(\mathbf{x}) = M\mathbf{x}. \tag{1.5.11}$$

**Example** Suppose $L : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$L(x, y, z) = (3x - 2y + z, 4x + y).$$

Then

$$L(\mathbf{e}_1) = L(1, 0, 0) = (3, 4),$$

$$L(\mathbf{e}_2) = L(0, 1, 0) = (-2, 1),$$

and

$$L(\mathbf{e}_3) = L(0, 0, 1) = (1, 0).$$

So if we let

$$M = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 1 & 0 \end{bmatrix},$$

then

$$L(x, y, z) = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

For example,

$$L(1, -1, 3) = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 + 2 + 3 \\ 4 - 1 + 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}.$$

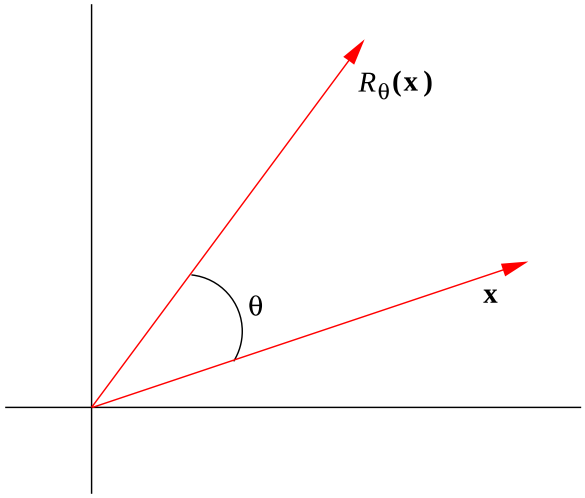


Figure 1.5.1 Rotating a vector in the plane

**Example** Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates a vector $\mathbf{x}$ in $\mathbb{R}^2$ counterclockwise through an angle $\theta$, as shown in Figure 1.5.1. Geometrically, it seems reasonable that $R_\theta$ is a linear function; that is, rotating the vector $\mathbf{x} + \mathbf{y}$ through an angle $\theta$ should give the same result as first rotating $\mathbf{x}$ and $\mathbf{y}$ separately through an angle $\theta$ and then adding, and rotating a vector $a\mathbf{x}$ through an angle $\theta$ should give the same result as first rotating $\mathbf{x}$ through an angle $\theta$ and then multiplying by $a$. Now, from the definition of $\cos(\theta)$ and $\sin(\theta)$,

$$R_\theta(\mathbf{e}_1) = R_\theta(1,0) = (\cos(\theta), \sin(\theta))$$

(see Figure 1.5.2), and, since $\mathbf{e}_2$ is $\mathbf{e}_1$ rotated, counterclockwise, through an angle $\frac{\pi}{2}$,

$$R_\theta(\mathbf{e}_2) = R_{\theta+\frac{\pi}{2}}(\mathbf{e}_1) = \left(\cos\left(\theta + \frac{\pi}{2}\right), \sin\left(\theta + \frac{\pi}{2}\right)\right) = (-\sin(\theta), \cos(\theta)).$$

Hence

$$R_\theta(x,y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \qquad (1.5.12)$$

You are asked in Problem 9 to verify that the linear function defined in (1.5.12) does in fact rotate vectors through an angle $\theta$ in the counterclockwise direction. Note that, for example, when $\theta = \frac{\pi}{2}$, we have

$$R_{\frac{\pi}{2}}(x,y) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In particular, note that $R_{\frac{\pi}{2}}(1,0) = (0,1)$ and $R_{\frac{\pi}{2}}(0,1) = (-1,0)$; that is, $R_{\frac{\pi}{2}}$ takes $\mathbf{e}_1$ to $\mathbf{e}_2$ and $\mathbf{e}_2$ to $-\mathbf{e}_1$. For another example, if $\theta = \frac{\pi}{6}$, then

$$R_{\frac{\pi}{6}}(x,y) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

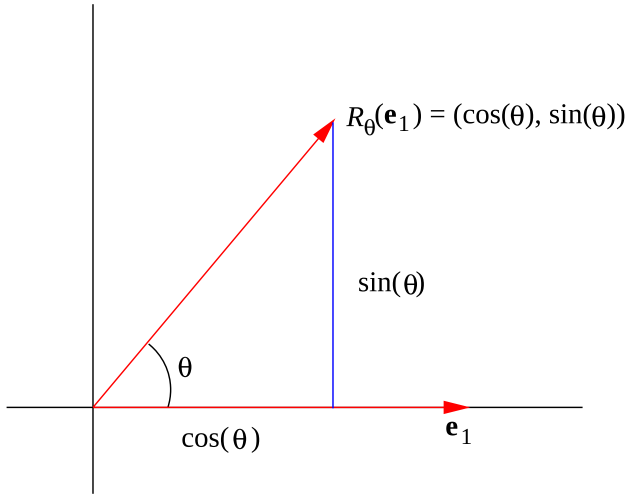


Figure 1.5.2 Rotating $\mathbf{e}_1$ through an angle $\theta$

In particular,

$$R_{\frac{\pi}{6}}(1,2) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} - 1 \\ \frac{1}{2} + \sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-2}{2} \\ \frac{1+2\sqrt{3}}{2} \end{bmatrix}.$$

### Affine functions

**Definition** We say a function $A : \mathbb{R}^m \to \mathbb{R}^n$ is *affine* if there is a linear function $L : \mathbb{R}^m \to \mathbb{R}^n$ and a vector $\mathbf{b}$ in $\mathbb{R}^n$ such that

$$A(\mathbf{x}) = L(\mathbf{x}) + \mathbf{b} \tag{1.5.13}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$.

An affine function is just a linear function plus a translation. From our knowledge of linear functions, it follows that if $A : \mathbb{R}^m \to \mathbb{R}^n$ is affine, then there is an $n \times m$ matrix $M$ and a vector $\mathbf{b}$ in $\mathbb{R}^n$ such that

$$A(\mathbf{x}) = M\mathbf{x} + \mathbf{b} \tag{1.5.14}$$

for all $\mathbf{x}$ in $\mathbb{R}^m$. In particular, if $f : \mathbb{R} \to \mathbb{R}$ is affine, then there are real numbers $m$ and $b$ such that

$$f(x) = mx + b \tag{1.5.15}$$

for all real numbers $x$.

**Example** The function

$$A(x, y) = (2x + 3, y - 4x + 1)$$

is an affine function from $\mathbb{R}^2$ to $\mathbb{R}^2$ since we may write it in the form

$$A(x,y) = L(x,y) + (3,1),$$

where $L$ is the linear function

$$L(x,y) = (2x, y - 4x).$$

Note that $L(1,0) = (2,-4)$ and $L(0,1) = (0,1)$, so we may also write $A$ in the form

$$A(x,y) = \begin{bmatrix} 2 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

**Example** The affine function

$$A(x,y) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

first rotates a vector, counterclockwise, in $\mathbb{R}^2$ through an angle of $\frac{\pi}{4}$ and then translates it by the vector $(1,2)$.

### Problems

1. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be vectors in $\mathbb{R}^m$ and define $L : \mathbb{R}^m \to \mathbb{R}^n$ by

   $$L(\mathbf{x}) = (\mathbf{a}_1 \cdot \mathbf{x}, \mathbf{a}_2 \cdot \mathbf{x}, \ldots, \mathbf{a}_n \cdot \mathbf{x}).$$

   Show that $L$ is linear. What does $L$ look like in the special cases
   (a) $m = n = 1$?
   (b) $n = 1$?
   (c) $m = 1$?

2. For each of the following functions $f$, find the dimension of the domain space, the dimension of the range space, and state whether the function is linear, affine, or neither.
   (a) $f(x,y) = (3x - y, 4x, x + y)$
   (b) $f(x,y) = (4x + 7y, 5xy)$
   (c) $f(x,y,z) = (3x + z, y - z, y - 2x)$
   (d) $f(x,y,z) = (3x - 4z, x + y + 2z)$
   (e) $f(x,y,z) = \left(3x + 5, y + z, \frac{1}{x + y + z}\right)$
   (f) $f(x,y) = 3x + y - 2$
   (g) $f(x) = (x, 3x)$
   (h) $f(w,x,y,z) = (3x, w + x - y + z - 5)$
   (i) $f(x,y) = (\sin(x + y), x + y)$
   (j) $f(x,y) = (x^2 + y^2, x - y, x^2 - y^2)$
   (k) $f(x,y,z) = (3x + 5, y + z, 3x - z + 6, z - 1)$

3. For each of the following linear functions $L$, find a matrix $M$ such that $L(\mathbf{x}) = M\mathbf{x}$.
   (a) $L(x, y) = (x + y, 2x - 3y)$
   (b) $L(w, x, y, z) = (x, y, z, w)$
   (c) $L(x) = (3x, x, 4x)$
   (d) $L(x) = -5x$
   (e) $L(x, y, z) = 4x - 3y + 2z$
   (f) $L(x, y, z) = (x + y + z, 3x - y, y + 2z)$
   (g) $L(x, y) = (2x, 3y, x + y, x - y, 2x - 3y)$
   (h) $L(x, y) = (x, y)$
   (i) $L(w, x, y, z) = (2w + x - y + 3z, w + 2x - 3z)$

4. For each of the following affine functions $A$, find a matrix $M$ and a vector $\mathbf{b}$ such that $A(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$.
   (a) $A(x, y) = (3x + 4y - 6, 2x + y - 3)$
   (b) $A(x) = 3x - 4$
   (c) $A(x, y, z) = (3x + y - 4, y - z + 1, 5)$
   (d) $A(w, x, y, z) = (1, 2, 3, 4)$
   (e) $A(x, y, z) = 3x - 4y + z - 1$
   (f) $A(x) = (3x, -x, 2)$
   (g) $A(x_1, x_2, x_3) = (x_1 - x_2 + 1, x_1 - x_3 + 1, x_2 + x_3)$

5. Multiply the following.
   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$
   (b) $\begin{bmatrix} -1 & 2 \\ 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 2 & 1 - 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

6. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that maps a vector $\mathbf{x} = (x, y)$ to its reflection across the horizontal axis. Find the matrix $M$ such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

7. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that maps a vector $\mathbf{x} = (x, y)$ to its reflection across the line $y = x$. Find the matrix $M$ such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that maps a vector $\mathbf{x} = (x, y)$ to its reflection across the line $y = -x$. Find the matrix $M$ such that $L(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

9. Let $R_\theta$ be defined as in (1.5.12).
   (a) Show that for any $\mathbf{x}$ in $\mathbb{R}^2$, $\|R_\theta(\mathbf{x})\| = \|\mathbf{x}\|$.
   (b) For any $\mathbf{x}$ in $\mathbb{R}^2$, let $\alpha$ be the angle between $\mathbf{x}$ and $R_\theta(\mathbf{x})$. Show that $\cos(\alpha) = \cos(\theta)$. Together with (a), this verifies that $R_\theta(\mathbf{x})$ is the rotation of $\mathbf{x}$ through an angle $\theta$.

10. Let $S_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function that rotates a vector $\mathbf{x}$ clockwise through an angle $\theta$. Find the matrix $M$ such that $S_\theta(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

11. Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, we call the set

$$\{\mathbf{y} : \mathbf{y} = f(\mathbf{x}) \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^m\}$$

the *image*, or *range*, of $f$.

(a) Suppose $L : \mathbb{R} \to \mathbb{R}^n$ is linear with $L(1) \neq \mathbf{0}$. Show that the image of $L$ is a line in $\mathbb{R}^n$ which passes through $\mathbf{0}$.

(b) Suppose $L : \mathbb{R}^2 \to \mathbb{R}^n$ is linear and $L(\mathbf{e}_1)$ and $L(\mathbf{e}_2)$ are linearly independent. Show that the image of $L$ is a plane in $\mathbb{R}^n$ which passes through $\mathbf{0}$.

12. Given a function $f : \mathbb{R}^m \to \mathbb{R}$, we call the set

$$\{(x_1, x_2, \ldots, x_m, x_{m+1}) : x_{m+1} = f(x_1, x_2, \ldots, x_m)\}$$

the graph of $f$. Show that if $L : \mathbb{R}^m \to \mathbb{R}$ is linear, then the graph of $L$ is a hyperplane in $\mathbb{R}^{m+1}$.